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DERIVATIONS OF SIMPLICIAL CONIC ALGEBRAS. APPLICATIONS TO DEFORMATIONS OF CURVES ON TORIC SURFACES

by

Jan Arthur Christophersen
University of Oslo
Norway

Introduction

A cone σ in $\mathbb{R}_{+}^{n}$ corresponds to a normal n-dimensional affine toric variety Spec $k[\Lambda]$. Here k is an algebraically closed field of characteristic 0, $\Lambda = \sigma \cap \mathbb{Z}^n$ and $k[\Lambda]$ is the semigroup ring of Λ over k. (See [0] chapter I.). To solve the moduli problem for such toric varieties, one must compute the André algebra cohomology $H^i(k,k[\Lambda];\ k[\Lambda])$ $(i=0,1,2)$. (See [L].) The first part of this paper includes an explicit description of $H^0(k,k[\Lambda];\ k[\Lambda]) = \mathrm{Der}_k(k[\Lambda])$ when Λ is <u>simplicial</u>.

If $n=2$, then all cones are simplicial. Let m_1 and m_2 generate σ and choose two positive integers m and n. The results of Part I are used in Part II to find all first order deformations of $\mathrm{Spec}(k[\Lambda]/(\underline{x}^{m\cdot m_1} + \underline{x}^{n\cdot m_2}))$, lying on the toric surface. (This notion: "deformations lying on", will be made precise.) The results generalize well known facts about quasi-homogeneous plane curves. The interest for such curves originated from trying to use Newton filtration methods ([K]) to deform affine plane curves.

Part I.

1. The generating paralellotope of a simplicial cone.

In this paper a subset $\sigma \subseteq \mathbb{R}^n_+$ is called a cone if there exist $m_1,\dots,m_s \in \mathbb{Z}^n_+$ such that $\sigma = \{\sum_{i=1}^{s} a_i m_i \mid a_1,\dots,a_n \in \mathbb{R}_+\}$. If the m_i are irredundant and each m_i is primitive, then they are uniquely determined by σ and are called the fundamental generators of σ. The cone σ is simplicial if its fundamental generators are $\mathbb{R}$ linearly independent. We will only regard simplicial cones with fundamental generators $m_1,\dots,m_n$ organized so that $\det(m_1,\dots,m_n) > 0$.

Let $\Lambda = \sigma \cap \mathbb{Z}^n_+$ be the sub-semigroup of $\mathbb{Z}^n_+$ associated to σ. Define P^* by

$$P^* = \{\sum_{i=1}^{n} \alpha_i m_i \mid \alpha_i \in \mathbb{Q} \text{ and } 0 \leqslant \alpha_i < 1\}.$$

The generating parallellotope of Λ is $P = P^* \cap \Lambda$. Any point in Λ can be uniquely written as

$$a + \sum_{i=1}^{n} r_i m_i$$

where $a \in P$ and the r_i are non-negative integers.

Consider the torus group $T = \mathbb{R}^n / \langle m_1,\dots,m_n\rangle$ and the injective map $v\colon P \to T$. There is a unique group structure on P such that v becomes a homomorphism, i.e the group addition is vector addition modulo $\langle m_1,\dots,m_n\rangle$. The order of P is $d = \det(m_1,\dots,m_n)$. If $a \in P$, set $v(a) = \bar{a}$ and $a^{-1} = v^{-1}(\bar{a}^{-1}) \in P$.

Each face of σ can be viewed as simplicial cone in a lower dimensional real space. Intersecting P with a face of σ gives thus a subgroup of P. Of course a^{-1} must be in the smallest face containing a. Define $\sigma_i: P \to \{0,1\}$, $i=1,\ldots,n$, by the relation

$$a + a^{-1} = \sum_{i=1}^{n} \sigma_i(a) \cdot m_i$$

The geometrical meaning of σ_i is the following: $(m_{i_1},\ldots,m_{i_k})$, $2 \leq k \leq n$, generate the smallest face of σ containing a if and only if $\sigma_i(a) = 1$ for $i \in \{i_1,\ldots,i_k\}$ and $\sigma_i(a) = 0$ for $i \in \{1,2,\ldots,n\}-\{i_1,\ldots,i_k\}$.

Define linear maps $\nu_i: \Lambda \to \mathbb{Z}_+$, $i=1,\ldots,n$, by

$$\nu_i(a) = \det(m_1,\ldots,m_{i-1},a, m_{i+1},\ldots,m_n).$$

Each ν_i induces a group homomorphism $\bar{\nu}_i: P \to \mathbb{Z}/\langle d\rangle$; $\bar{\nu}_i(a) = \nu_i(a) \pmod{d}$. Notice that $\nu_i(a) + \nu_i(a^{-1}) = \sigma_i(a)\cdot d$, so $\nu_i(a)=0$ if and only if $\sigma_i(a) = 0$. It follows from the definition of P that for all $a \in P$

$$a = \sum_{i=1}^{n} \frac{\nu_i(a)}{d} \cdot m_i$$

Since $\bar{\nu}_i^{-1}(0)$ are the elements of P which are in the span of $m_1,\ldots,m_{i-1}, m_{i+1},\ldots,m_n$, P is cyclic if there exists a codimension 1 face of P^* that has no integral points other than 0.

If $\bar{a} + \bar{b} = \bar{c}$, there must be $\rho_i(a,b) \in \{0,1\}$ such that

$$a + b = c + \sum_{i=1}^{n} \rho_i(a,b)m_i$$

These relations define maps $\rho_i: P \times P \to \{0,1\}$, $i=1,\ldots,n$. Applying ν_i to the above relation we find for all $i=1,\ldots,n$

$$\nu_i(a) + \nu_i(b) = \nu_i(c) + \rho_i(a,b)\cdot d$$

So $\rho_i(a,b) = 0$ if $\nu_i(a) \leqslant \nu_i(c)$ or $\nu_i(b) \leqslant \nu_i(c)$, and $\rho_i(a,b) = 1$ otherwise.

We can also give Λ a partial ordering: $\lambda_1 < \lambda_2$ if there exists $\mu \in \Lambda$ such that $\lambda_1 + \mu = \lambda_2$. A minimal element of Λ will always mean a minimal element of $\Lambda_+ = \Lambda - \{0\}$. All the minimal elements of Λ, except the fundamental generators, are in P.

2. The semigroup algebra $k[\Lambda]$.

Let

$$k[\Lambda] = \{\sum \alpha_\lambda \underline{x}^\lambda \in k[x_1, \ldots, x_n] \mid \lambda \in \Lambda,\ \alpha_\lambda \in k\}$$

be the semigroup algebra of Λ over k. It is a conic ring in the sense of [K]. If $a \in P$ set

$$u_a = \underline{x}^a, (u_0 = 1), \text{ and } z_i = \underline{x}^{m_i}.$$

Give indices $i=0,1,\ldots,d-1$ such that $P = \{a_0, a_1, \ldots, a_{d-1}\}$ and $a_0 = 0$. The preceeding remarks show that

$$k[\Lambda] = k[u_{a_1}, u_{a_2}, \ldots, u_{a_{d-1}}, z_1, \ldots, z_n]$$

and that all monomials in $k[\Lambda]$ can be uniquely written as $u_a \prod_{i=1}^{n} z_i^{r_i}$ for an $a \in P$ and non-negative integers r_i. We call this the <u>normal form</u> of a monomial in $k[\Lambda]$. This shows that $k[\Lambda]$ is finitely generated as a $k[z_1, \ldots, z_n]$ module. Notice that $k[z_1, \ldots, z_n]$ is a free k algebra.

If $k[U_1, \ldots, U_{d-1}, Z_1, \ldots, Z_n]$ is the free k algebra with $d-1+n$ indeterminates, we have a natural epimorphism $\phi\colon k[\underline{U}, \underline{Z}] \to k[\Lambda]$; $\phi(U_i) = u_{a_i}$, $\phi(Z_i) = z_i$. (Sometimes it will be more convenient to write U_a for U_i if $a = a_i$.)

$$f = \prod_{i=1}^{d-1} U_i^{r_i} - U_{i_0} \prod_{i=1}^{n} Z_i^{s_i}$$

where $\sum_{i=1}^{d-1} r_i a_i = a_{i_0} + \sum_{i=1}^{n} s_i m_i$. We continue with induction on $N = \sum_{i=1}^{d-1} r_i$. If $N=2$ then f is obviously one of the $F_{a,b}$. Again let k be the smallest integer such that $r_k \neq 0$, and find $a_{j_0} \in P$ and non negative integers $\gamma_1, \dots, \gamma_n$ such that

$$(r_k - 1)a_k + \sum_{i=k+1}^{d-1} r_i a_i = a_{j_0} + \sum_{i=1}^{n} \gamma_i m_i$$

Then

$$f = U_k\left(U_k^{r_k-1} \prod_{i=k+1}^{d-1} U_i^{r_i} - U_{j_0} \prod_{i=1}^{n} Z_i^{\gamma_i}\right) + \prod_{i=1}^{n} Z_i^{\gamma_i} F_{a_k, a_{j_0}}$$

and the proposition is proved. □

This presentation of $k[\Lambda]$ involves many redundant indeterminants U_i and generators of $\ker\phi$, but it does give a normal form for the generating relations in $k[\Lambda]$ and as such is appropriate for our purposes.

3. Derivations of $k[\Lambda]$.

If $f \in k[\Lambda]$ then f can be written as $f = \sum_{i=1}^{d-1} u_{a_i} f_i(z_1, \dots, z_n)$ where $f_i(z_1, \dots, z_n) \in k[z_1, \dots, z_n]$. Take the represenative $F = \sum_{i=1}^{d-1} U_i f_i(Z_1, \dots, Z_n) \in k[\underline{U}, \underline{Z}]$ and write $f_{a_i} = \phi(\frac{\partial F}{\partial U_i})$ and $f_{z_i} = \phi(\frac{\partial F}{\partial Z_i})$.

Proposition 2. Let M be a $k[\Lambda]$ module and suppose no monomial in $k[\Lambda]$ is a zero divisor for M. A k-linear map $D: k[\Lambda] \to M$ is a derivation (i.e. $D \in \mathrm{Der}_k(k[\Lambda], M)$) if and only if

i) $D(f) = \sum_{i=1}^{d-1} f_{a_i} D(u_{a_i}) + \sum_{i=1}^{n} f_{z_i} D(z_i)$ for all $f \in k[\Lambda]$

and

ii) $d\, u_{a^{-1}} D(u_a) = \sum_{i=1}^{n} \nu_i(a) \prod_{j \neq i} z_j^{\sigma_j(a)} D(z_i)$ for all $a \in P-\{0\}$.

Proof. Assume first that $D \in \mathrm{Der}_k(k[\Lambda],M)$. Condition i) is straight from the definition of a derivaiton. Recall that $da = \sum_{i=1}^{n} \nu_i(a)m_i$ for $a \in P$, so $u_a^d = \prod_{i=1}^{n} z_i^{\nu_i(a)}$. Apply the derivation D and obtain,

$$d\, u_a^{d-1} D(u_a) = \sum_{i=1}^{n} \nu_i(a) \Big(\prod_{j \neq i} z_j^{\nu_j(a)} \Big) z_i^{\nu_i(a)-1} D(z_i)$$

The right side makes sense since $\nu_i(a)-1 < 0$ if and only if $\nu_i(a) = 0$. Combining the functions ν_i and σ_i we find

$$(d-1)a = a^{-1} + \sum_{i=1}^{n} (\nu_i(a)-\sigma_i(a))m_i .$$

Since $\nu_i(a) - \sigma_i(a) \geq 0$ for all $a \in P$, we have

$$du_{a^{-1}} \prod_{i=1}^{n} z_i^{\nu_i(a)-\sigma_i(a)} D(u_a) = \sum_{i=1}^{n} \nu_i(a) z_i^{\nu_i(a)-1} \prod_{j \neq i} z_j^{\nu_j(a)} D(z_i).$$

Since $\nu_i(a) -1 - (\nu_i(a)-\sigma_i(a)) = \sigma_i(a) - 1 < 0$ only if $\nu_i(a) = 0$, we can factor out $\prod_{i=1}^{n} z^{\nu_i(a)-\sigma_i(a)}$ and get condition ii).

Assume that D satisfies i) and ii). We must prove that D is well defined, i.e. that

$$D(u_a u_b) = D(u_c \prod_{i=1}^{n} z_i^{\rho_i(a,b)})$$

where $\bar{a} + \bar{b} = \bar{c}$, since Proposition 1 says that these are the generating relations in $k[\Lambda]$. Choose a and b in $P-\{0\}$ and set $\rho_i = \rho_i(a,b)$. Now

$$(*)\quad D(u_a u_b - u_c \Pi z_i^{\rho_i}) = u_a D(u_b) + u_b D(u_a) - D(u_c)\Pi z_i^{\rho_i} - (\sum_i \rho_i u_c \prod_{j\neq i} z^{\rho_j} D(z_i))$$

since $\rho_i - 1 < 0$ if and only if $\rho_i = 0$.

Notice that $c^{-1} + a = b^{-1} + \sum_{i=1}^{n} (\rho_i + \sigma_i(c) - \sigma_i(b))m_i$ and that $\rho_i + \sigma_i(c) - \sigma_i(b) > 0$. Multiplying $(*)$ with $du_{c^{-1}}$ then gives us

$$d\,u_{b^{-1}} \prod_i z_i^{\rho_i+\sigma_i(c)-\sigma_i(b)} D(u_b) + d\,u_{a^{-1}} \prod_i z_i^{\rho_i+\sigma_i(c)-\sigma_i(a)} D(u_a)$$

$$- d\,u_{c^{-1}} \prod_i z_i^{\rho_i} D(u_c) - \sum_i \rho_i d \prod_{j\neq i} z_j^{\rho_j} \prod_{j=1}^{n} z_j^{\sigma_j(c)} D(z_i)$$

Using condition ii) this becomes

$$\sum_i \prod_{j\neq i} z_j^{\rho_j+\sigma_i(c)} D(z_i)(\nu_i(b) z_i^{\rho_i+\sigma_i(c)-\sigma_i(b)} + \nu_i(a)\, z_i^{\rho_i+\sigma_i(c)-\sigma_i(a)}$$

$$\cdot \qquad - \nu_i(c) z_i^{\rho_i+\sigma_i(c)} - \rho_i d\, z_i^{\sigma_i(c)})$$

since $\nu_i(a) = 0$ if and only if $\sigma_i(a) = 0$ we must have $\nu_i(a) z_i^{\rho_i+\sigma_i(c)-\sigma_i(a)} = \nu_i(a) z_i^{\rho_i+\sigma_i(c)-1}$, $\nu_i(b) z_i^{\rho_i+\sigma_i(c)-\sigma_i(b)} = \nu_i(b) z_i^{\rho_i+\sigma_i(c)-1}$ and $\nu_i(c) z_i^{\rho_i+\sigma_i(c)} = \nu_i(c) z_i^{\rho_i+\sigma_i(c)-1}$. We can therefore factor $z_i^{\rho_i+\sigma_i(c)-1}$ out of the parenthesis in the i'th summand and are left with

$$(**)\qquad \nu_i(b) + \nu_i(a) - \nu_i(c) - \rho_i d\, z_i^{1-\rho_i}$$

If $\rho_i = 0$ then $\rho_i d\, z_i^{1-\rho_i} = \rho_i d$ and if $\rho_i = 1$ then $\rho_i d z_i^{1-\rho_i} = \rho_i \cdot d$, so $(**)$ equals $\nu_i(b) + \nu_i(a) - \nu_i(c) - \rho_i d$ which we recall is zero. □

Since $k[\Lambda]$ is an integral domain and char $k=0$, $\mathrm{Der}_k(k[\Lambda])$ is isomorphic to a submodule of $\bigoplus_{i=1}^{n} k[\Lambda]$. (One shows that $D \to (D(z_1), D(z_2), \ldots, D(z_n))$ is injective, remembering that $k[\Lambda]$ is integral over $k[z_1, \ldots, z_n]$.) We shall give an explicit description of this submodule.

When the module M of Proposition 2 is $k[\Lambda]$ itself, then condition ii) reduces to

$$\sum_i \nu_i(a) \prod_{j \neq i} z_j^{\sigma_j(a)} D(z_i) \in (u_{a^{-1}})$$

for all $a \in P-\{0\}$.

<u>Lemma 1</u>. Let $\mathfrak{a}_{i,a} \subseteq k[\Lambda]$ be the ideal generated by z_i and those u_b such that $\nu_i(b) > \nu_i(a)$. If $A_i \in k[\Lambda]$, $i=1,\ldots,n$, then

$$\sum_{i=1}^{n} \nu_i(a) \prod_{j \neq i} z_j^{\sigma_j(a)} A_i \in (u_{a^{-1}})$$

if and only if

$$A_i \in \begin{cases} (1) & \text{if } \nu_i(a) = 0 \\ \mathfrak{a}_{i,a^{-1}} & \text{if } \nu_i(a) > 0 \end{cases}$$

for all $i=1,\ldots,n$.

<u>Proof</u>. First we prove that $\sum_{i=1}^{n} \nu_i(a) \prod_{j \neq i} z_j^{\sigma_j(a)} A_i \in (u_{a^{-1}})$ if and only if $B_i = \nu_i(a) \prod_{j \neq i} z_j^{\sigma_j(a)} A_i \in (u_{a^{-1}})$ for all $i=1,\ldots,n$. Assume that the sum is an element of $(u_{a^{-1}})$, but that for example B_k is not. Then $\sigma_k(a) \neq 0$ and B_k has a monomial

$u_b \prod_{j=1}^{n} z_j^{r_j} \prod_{j\neq i} z_j^{\sigma_j(a)}$ which is not in $(u_{a^{-1}})$. We must have $r_k = 0$ since $u_{a^{-1}} u_a = \prod_{j=1}^{n} z_j^{\sigma_j(a)} \in (u_{a^{-1}})$. This monomial must be killed by some other B_i, but that is impossible since z_k is a factor in all other B_i where $\nu_i(a) > 0$.

It is therefore enough to check the condition of the lemma for each B_i. Fix a $k \in \{1,\ldots,n\}$. If $\nu_i(a) = 0$ then of course A_k can be anything so assume that $\nu_i(a) > 0$ (and therefore $\sigma_k(a)=1$).

Of course $z_k \prod_{j\neq k} z_j^{\sigma_j(c)} = \prod_{j=1}^{n} z_j^{\sigma_j(c)} = u_{a^{-1}} u_a \in (u_{a^{-1}})$. If $b \in P$ and $\nu_k(b) > \nu_k(a^{-1})$, then we can find $c \in P$ such that $\bar{a}^{-1} + \bar{c} = \bar{b}$ and $\rho_k(a^{-1},c) = 0$. Now $\sigma_j(a) - \rho_j(a^{-1},c) \geq 0$, since $\sigma_j(a) = 0$ implies $\rho_j(a^{-1},c) = 0$. Thus

$$u_b \prod_{j\neq k} z_j^{\sigma_j(a)} = u_{a^{-1}} u_c \prod_{j\neq k} z_j^{\sigma_j(a)-\rho_j(a^{-1},c)} \in (u_{a^{-1}}).$$

It follows that if $A_k \in \mathfrak{a}_{k,a^{-1}}$ then $B_k \in (u_{a^{-1}})$.

If $B_k \in (u_{a^{-1}})$ and $A_k \notin \mathfrak{a}_{k,a^{-1}}$ then A_k must have a monomial $u_b \prod z_i^{r_i}$ with $r_k = 0$ and $\nu_k(b) < \nu_k(a^{-1})$. There must be a monomial $g \in k[\Lambda]$ such that

$$u_b \prod_{j\neq k} z_j^{r_j+\sigma_j(a)} = g u_{a^{-1}}$$

But since the normal form for monomials in $k[\Lambda]$ is unique, $g = u_c \prod z_i^{s_i}$ where $\bar{a}^{-1} + \bar{c} = \bar{b}$ and the s_i are nonnegative integers. Now $\rho_k(a^{-1},c) = 1$ since $\nu_k(b) < \nu_k(a^{-1})$ and z_k is a factor in $g u_{a^{-1}}$. Contradiction. □

Summing up Proposition 2 and Lemma 1 we have proved the following

<u>Theorem</u>. $\mathrm{Der}_k(k[\Lambda])$ is isomorphic to the submodule

$$\bigoplus_{i=1}^{n} \Big(\bigcap_{\substack{a\in P \\ v_i(a)\neq 0}} \mathfrak{a}_{i,a} \Big)$$

of $(k[\Lambda])^n$.

<u>Proof</u>. Since $\sum_i v_i(a) \prod_{j\neq i} z_j^{\sigma_j(a)} D(z_i) \in (u_{a^{-1}})$ for all $a \in P-\{0\}$, $D(z_i) \in \bigcap_{\substack{a\in P \\ v_i(a)\neq 0}} \mathfrak{a}_{i,a^{-1}}$. But since $v_i(a) = 0$ if and only if $v_i(a^{-1}) = 0$ we can exchange a for a^{-1}. □

<u>Part II</u>

<u>1. Deformations of subschemes</u>

A detailed and rigid treatment of the following informal remarks can be found in [L] chapter 4. Let Y be an affine scheme over a field k (k algebraically closed) and X a subscheme of Y. Let R be a local Artin k algebra. A <u>deformation of X on Y to R</u> is a commutative diagram

$$\begin{tikzcd}
X_R \arrow[rr, "\alpha"] \arrow[dr] & & Y\times \mathrm{Spec}(R) \arrow[dl] \\
& \mathrm{Spec}\ R & \\
X \arrow[uu] \arrow[rr, "i"] & & Y \arrow[uu, "p"'] \\
& \mathrm{Spec}\ k \arrow[uu] &
\end{tikzcd}$$

where i is the inclusion, p is the canonical imbedding, α is <u>any</u> closed imbedding and X_R is a deformation of X to R (i.e. $X_R \times_{\mathrm{Spec}\ R} \mathrm{Spec}\ k \simeq X$ and X_R flat over Spec R). Two such deformations X_R and X'_R are isomorphic if there exists an isomorphism $X_R \to X'_R$ inducing the identity on X. Define

$$\mathrm{Def}_{\{X\subseteq Y\}}(R) = \{\text{isomorphism classes of deformations of } X \text{ on } Y \text{ to } R\}$$

Suppose now that $X = \mathrm{Spec}\ A$, $Y = \mathrm{Spec}\ B$ and $A=B/(f)$ for some $f\in B$. Following Laudal [L] the deformations of A as a B-algebra to $k[\varepsilon]$ are classified by $H^1(B,A;A)$. Now $H^1(B,A;A) \simeq \mathrm{Hom}_B((f),A)$ which is isomorphic to A by associating $\xi \in \mathrm{Hom}_B((f),A)$ with $\xi(f)$. In our case, two deformations X_R and X'_R are isomorphic even though

$$\begin{tikzcd}[column sep=small] & Y \times \mathrm{Spec}\ R & \\ X_R \arrow[ur, "\alpha"] \arrow[rr, "\sim"] & & X'_R \arrow[ul, "\alpha'"'] \end{tikzcd}$$

does not commute. When $R = k[\varepsilon]$ such isomorphisms are measured exactly by $\mathrm{Der}_k(B,A)$. Identifying $H^1(B,A;A)$ with A, we aquire the module classifying $\mathrm{Def}_{\{X\subseteq Y\}}(k[\varepsilon])$:

$$A/\{D(f)\,|\,D \in \mathrm{Der}_k(B,A)\}$$

Notice that when $B = k[\underline{x}]$ and $A = k[\underline{x}]/(f)$ we get the well known T^1 for hypersurfaces, $k[\underline{x}]/(f,\frac{\partial f}{\partial x_1},\ldots,\frac{\partial f}{\partial x_n})$.

2. The curves $C_{m,n}$ on a toric surface

We restrict our attention from now on to 2 dimensional cones in $\mathbb{R}^2_+$. All such cones are simplicial. What's more, the group P is cyclic (both $\bar{\nu}_1$ and $\bar{\nu}_2$ are isomorphisms) and $\sigma_1(a)=\sigma_2(a)=1$ for all $a\neq 0$ in P. The ring $k[\Lambda]$ is a complete intersection if and only if $P-\{0\}$ has only one minimal element ([R] Cor. to Satz 4). Here is a list of easily proven observations which will come in handy later.

<u>Lemma 2</u>

i) $a<b \iff \nu_i(a) < \nu_i(b)$, $i=1,2$, $a,b \in P$

ii) $\nu_i(a) = d - \nu_i(a^{-1})$, $i=1,2$

iii) $a^{-1} < b \iff a > b^{-1}$ if $a,b \in P-\{0\}$. a is maximal in $P \iff a^{-1}$ is minimal

iv) a and b lie on the same line through the origin if and only if $\nu_1(a)\nu_2(b) = \nu_2(a)\nu_1(b)$

v) P has only one minimal element if and only if all the elements of P lie on the same line through the origin. □

Let $\hat{a}$ be the member of P such that $\nu_1(\hat{a}) = d-1$, and let $\check{a}$ be such that $\nu_2(\check{a}) = d-1$. The ideals in the theorem of Part I become in the 2 dimensional case:

$$\bigcap_{a\in P} \mathfrak{a}_{1,a} = (z_1, u_{\hat{a}}) \quad \text{and} \quad \bigcap_{a\in P} \mathfrak{a}_{2,a} = (z_2, u_{\check{a}})$$

In other words if $D \in \mathrm{Der}_k(k[\Lambda])$ then $D(z_1) \in (z_1, u_{\hat{a}})$ and $D(z_2) \in (z_2, u_{\check{a}})$.

For any pair of positive integers (m,n) let $f_{m,n} = z_1^m + z_2^n$, $A_{m,n} = k[\Lambda]/(f_{m,n})$ and $C_{m,n} = \mathrm{Spec}\, A_{m,n}$. Set $X = \mathrm{Spec}\, k[\Lambda]$, the toric surface corresponding to $k[\Lambda]$. We are going to compute $\mathrm{Def}_{\{(C_{m,n} \subseteq X\}}(k[\varepsilon])$; that is - find a k basis for

$$A_{m,n} / \{D(f_{m,n}) \mid D \in \mathrm{Der}_k(k[\Lambda], A_{m,n})\}$$

For each pair (m,n) there is a diagonal grading of $k[\Lambda]$ determined by giving z_1 degree nd, z_2 degree md and u_a degree $m\nu_2(a) + n\nu_1(a)$ for $a \in P$. Thus $f_{m,n}$ is homogeneous of degree mnd and $C_{m,n}$ is an example of a quasi-homogeneous curve that is not in general a complete intersection.

3. Deformatons of $C_{m,n}$ on X

If $g \in k[\Lambda]$, denote by $\bar{g}$ its image in $A_{m,n}$. Notice that $A_{m,n}$ satisfies the restriction on the module M of Proposition 2, so we have

$$\{D(f_{m,n}) \mid D \in \mathrm{Der}_k(k[\Lambda],\ A_{m,n})\} =$$

$$\{m\bar{z}_1^{\,m-1}\bar{\alpha}_1+n\bar{z}_2^{\,n-1}\bar{\alpha}_2 \mid \nu_1(a)\bar{z}_2\bar{\alpha}_1+\ \nu_2(a)\bar{z}_1\bar{\alpha}_2 \in (\bar{u}_{a^{-1}})$$
$$\text{for all } a\in P,\ \alpha_1,\alpha_2 \in k[\Lambda]\}.$$

Since $\alpha_1 = \frac{1}{m}z_1$ and $\alpha_2 = \frac{1}{n}z_2$ always define derivations,and $f_{m,n} = mz_1^{m-1}(\frac{1}{m}z_1) + nz_2^{n-1}(\frac{1}{n}z_2)$, we can lift to $k[\Lambda]$. That is

$$A_{m,n}/\{D(f_{m,n}) \mid D \in \mathrm{Der}_k(k[\Lambda],A_{m,n})\} \simeq k[\Lambda]/\mathrm{Der}_{m,n}$$

where $\mathrm{Der}_{m,n} = \{mz_1^{m-1}\alpha_1+nz_2^{n-1}\alpha_2 \mid \nu_1(a)\bar{z}_2\bar{\alpha}_1+\nu_2(a)\bar{z}_1\bar{\alpha}_2 \in (\bar{u}_{a^{-1}})$ for all $a\in P$, $\alpha_1,\alpha_2 \in k[\Lambda]\}$. One checks that $\mathrm{Der}_{m,n}$ is an ideal in $k[\Lambda]$.

<u>Definition</u>. An element a in $P-\{0\}$ satisfies <u>condition L</u> if all $b < a^{-1}$ lie on the same line throught the origin.

All maximal elements of P satisfy condition L. If $k[\Lambda]$ is a complete intersection (i.e. P has only one minimal element), then all elements of $P-\{0\}$ satisfy condition L.

<u>Proposition 3</u>. If $k[\Lambda]$ is not a complete intersection then $\mathrm{Der}_{m,n}$ is generated as an ideal by: z_1^m, z_2^n, $u_{\hat{a}}z_1^{m-1}$, $u_{\check{a}}z_2^{n-1}$ and the monomials $u_a z_1^{m-1}z_2^{n-1}$ where a satisfies condition L.

If $k[\Lambda]$ is a complete intersection then $\mathrm{Der}_{m,n}$ is generated by z_1^m, z_2^n, $u_{\hat{a}} z_1^{m-1}$, $u_{\check{a}} z_2^{n-1}$. and $z_1^{m-1} z_2^{n-1}$

Proof. First assume $k[\Lambda]$ is not a complete intersection and let I be the ideal generated by the monomials mentioned in the proposition. To prove that $I \subseteq \mathrm{Der}_{m,n}$ we must exhibit choices of α_1 and α_2 that satisfy the conditon in the definition of $\mathrm{Der}_{m,n}$ and such that $m z_1^{m-1}\alpha_1 + n z_2^{n-1}$ equals one of the generators of I. The following table gives these choices

	To obtain	Choose	
1	z_1^m	$\alpha_1 = \frac{1}{m} z_1$	$\alpha_2 = 0$
2	z_2^n	$\alpha_1 = 0$	$\alpha_2 = \frac{1}{n} z_2$
3	$u_{\hat{a}} z_1^{m-1}$	$\alpha_1 = \frac{1}{m} u_{\hat{a}}$	$\alpha_2 = 0$
4	$u_{\check{a}} z_2^{n-1}$	$\alpha_1 = 0$	$\alpha_2 = \frac{1}{n} u_{\check{a}}$
5	$u_a z_1^{m-1} z_2^{n-1}$ a satisfies condition L	$\alpha_1 = \gamma_1 u_a z_2^{n-1}$ where $\gamma_1 = \dfrac{\nu_2(b)}{m\nu_2(b)+n\nu_1(b)}$	$\alpha_2 = \gamma_2 u_a z_1^{m-1}$ $\gamma_2 = \dfrac{\nu_1(b)}{m\nu_1(b)+n\nu_2(b)}$ for all $b < a^{-1}$.

Clearly the choices give us the monomial we wish to obtain. We must verify that $\nu_1(c)\bar{z}_2\bar{\alpha}_1 + \nu_2(c)\bar{z}_1\bar{\alpha}_2 \in (\bar{u}_{c^{-1}})$ for all $c \in P-\{0\}$ in each case. Cases 1 to 4 are just consequences of the theorem. To prove 5 notice first that if $\nu_1(b)\cdot\nu_2(c) = \nu_1(c)\cdot\nu_2(b)$ then

$$\frac{\nu_2(b)}{m\nu_2(b)+n\nu_1(b)} = \frac{\nu_2(c)}{m\nu_2(b)+n\nu_1(b)} \quad \text{and} \quad \frac{\nu_1(b)}{m\nu_2(b)+n\nu_1(b)} = \frac{\nu_1(c)}{m\nu_2(c)+n\nu_1(c)}$$

So since a satisfies condition L, Lemma 2 iv) shows that the definition of γ_1 and γ_2 makes sense. For $c \in P$, set $\bar{g}_c = \nu_1(c)\bar{z}_2\bar{\alpha}_1 + \nu_2(a)\bar{z}_1\bar{\alpha}_2$. Since $\bar{z}_1^m = -\bar{z}_2^n$

$$\bar{g}_c = \bar{u}_a\bar{z}_1^m(\gamma_1\nu_1(c)-\gamma_2\nu_2(c)) = \bar{u}_a\bar{z}_2^n(\gamma_2\nu_2(c)-\gamma_1\nu_1(c))$$

If $a < c^{-1}$, then $a^{-1} > c$ (Lemma 2 iii)) so $\gamma_2\nu_2(c) - \gamma_1\nu_1(c)=0$ and $\bar{g}_c \in (\bar{u}_{c^{-1}})$. If $a < c^{-1}$ then Lemma 2 i) says that $\nu_1(a) > \nu_1(c^{-1})$ or $\nu_2(a) > \nu_2(c^{-1})$. In either case Lemma 1 shows that $\bar{g}_c \in (\bar{u}_{c^{-1}})$.

Suppose now that $g \in Der_{m,n}$ while $g \notin I$. Then g has a monomial $u_a z_1^r z_2^s$ that is not in I, so $r<m$ and $s<n$. On the other hand $Der_{m,n} \subseteq (z_1^{m-1}, z_2^{n-1})$ so either $r=m-1$ or $s=n-1$. If $r=m-1$, then there must exist $\alpha_1, \alpha_2 \in k[\Lambda]$ such that

(i) $\alpha_1 = \beta_1 u_a z_2^s + \alpha_1'$ and $\alpha_2 = \beta_2 u_a z_1^{m-1} z_2^{n-1} + \alpha_2'$

for some $\beta_i \in k$, $\alpha_i' \in k[\Lambda]$ ($\beta_2=0$ if $s \neq n-1$)

(ii) for all $b \in P$

$$\nu_1(b)\beta_1 u_a z_2^{s+1} + \nu_2(b)\beta_2 u_a z_1^m z_2^{s-n+1} + \nu_1(b)z_2\alpha_1' + \nu_2(b)z_1\alpha_2 + h_b(z_1^m+z_1^n) \in (u_{b^{-1}})$$

for some $h_b \in k[\Lambda]$.

There must be a b with $\nu_1(a) < \nu_1(b^{-1})$ for otherwize $u_a z_1^r z_2^s \in (u_{\hat{a}} z_1^{m-1})$. For such b, $\nu_1(b)\beta_1 u_a z_2^{s+1}$ must be killed by $h_b z_2^n$ in (ii). This implies that $s=n-1$, so there must exist $c \in P$ such that $\nu_1(a) < \nu_1(c^{-1})$ and $\nu_2(a) < \nu_2(c^{-1})$. (Otherwise a would be maximal and $u_a z_1^r z_2^s \in I$ by Lemma 2 iii))

That means that $u_a z_2^{n-1} \notin \mathfrak{a}_{1,c^{-1}}$ and $u_a z_1^{m-1} \notin \mathfrak{a}_{2,c^{-1}}$. At the same time Lemma 1 says that

$$\nu_1(c)\beta_1 u_a z_2^{n-1} + \nu_1(c)\alpha_1' + h_c z_2^{n-1} \in \mathfrak{a}_{1,c^{-1}}$$

$$\nu_2(c)\beta_2 u_a z_1^{m-1} + \nu_2(c)\alpha_2' + h_c z_1^{m-1} \in \mathfrak{a}_{2,c^{-1}}$$

Therefore $\nu_1(c)\beta_1 u_a z_2^{n-1}$ (resp. $\nu_2(c)\beta_2 u_a z_1^{m-1}$) must be killed by $h_c z_2^{n-1}$ (resp. $h_c z_1^{m-1}$). But then $h_c = -\beta_1\nu_1(c)u_a + h_c' = -\beta_2\nu_2(a)u_a + h_c''$ and u_a is not a monomial in h_c' or h_c''. So $\beta_1\nu_1(c) = \beta_2\nu_2(c)$ for all c such that $a < c^{-1}$. That means that a satisfies condition L if $a \neq 0$, so $a=0$. But if $a=0$, then the argument above entails that all points in P must lie on the same line through the origin contradicting the assumption. (Lemma 2v)). The same argument works if we start with $s=n-1$.

If $k[\Lambda]$ is a complete intersection set

$$\alpha_1 = \frac{\nu_2(c)}{m\nu_2(c)+n\nu_1(c)}\, z_2^{n-1} \quad \text{and} \quad \alpha_2 = \frac{\nu_1(c)}{m\nu_2(c)+n\nu_1(c)}\, z_1^{m-1}$$

The coefficients are constant for all $c \in P-\{0\}$ so $z_1^{m-1} z_2^{n-1} \in \mathrm{Der}_{m,n}$. The argument above proves also in this case that $\mathrm{Der}_{m,n} \subseteq (z_1^m,\ z_2^n,\ z_1^{m-1}z_2^{n-1},\ u_{\hat{a}} z_1^{m-1},\ u_{\check{a}} z_2^{n-1})$. □

We acquire immediately a description of $\mathrm{Def}_{\{C_{m,n}\subseteq X\}}(k[\varepsilon])$.

<u>Corollary 1</u>. The tangent space of $\mathrm{Def}_{\{C_{m,n}\subseteq X\}}$, $\mathrm{Def}_{\{C_{m,n}\subseteq X\}}(k[\varepsilon]) \simeq k[\Lambda]/\mathrm{Der}_{m,n}$, has a monomial k vector space basis consisting of 1 and the monomials of $k[\Lambda]$ that are not among the following:

monomials in (z_1^m, z_2^n)

$u_{\hat{a}} z_1^{m-1} z_2^s$, $s = 0,1,\dots,n-2$ (None if $n=1$)

$u_{\check{a}} z_1^r z_2^{n-1}$, $r = 0,1,\dots,m-2$ (None if $m=1$)

$u_a z_1^{m-1} z_2^{n-1}$ where a satisfies condition L.

$z_1^{m-1} z_2^{n-1}$ if $k[\Lambda]$ is a complete intersection

□

<u>Definition</u>. Define δ to be the number of elements a in $P-\{0\}$ such that all elements of P that are smaller then a lie on the same line through the origin.

Using Lemma 2 one sees easily that δ also equals the number of elements that satisfy condition L.

<u>Corollary 2</u>. The dimension of the tangent space of

$\mathrm{Def}_{\{C_{m,n} \subseteq X\}}(k[\varepsilon])$ is

$mnd - m - n - d + 2$ if $k[\Lambda]$ is a complete intersection and

$mnd - m - n - \delta + 2$ if $k[\Lambda]$ is not a complete intersection.

<u>Proof</u>. Just count up the basis. □

<u>4. Remarks and examples</u>

If S is the area of the triangle $0, m\cdot m_1, n\cdot m_2$, then $mnd = 2S$. The formulas in Corollary 2 are similar to the formula for the Milnor number of plane affine curves in [K]. (When $\Lambda = \mathbb{Z}_+^2$

then $\dim_k k[\Lambda]/\mathrm{Der}_{m,n} = \dim_k k[x,y]/(x^{m-1}, y^{m-1}) = mn-m-n-1+2 = (m-1)(n-1)$ as it should be.)

If $k[\Lambda]$ is a complete intersection, then we can use the above results to compute the Milnor number $\mu(C_{m,n})$ as defined in [B & G]. When Λ has only one minimal element one easily obtains

$$k[\Lambda] \simeq k[x,y,z]/(z^d - xy)$$

so $\tau_X = \dim_k H^1(k, k[\Lambda];\ k[\Lambda]) = d-1$. On the other hand since $C_{m,n}$ also is a complete intersection, the short exact sequence

$$0 \to (f_{m,n}) \to k[\Lambda] \to A_{m,n} \to 0$$

gives us an exact sequence of algebra cohomology ([L]) which in this case reduces to

$$0 \to k[\Lambda]/\mathrm{Der}_{m,n} \to H^1(k, A_{m,n}; A_{m,n}) \to H^1(k, k[\Lambda]; k[\Lambda]) \to 0.$$

Therefore $\tau_{C_{m,n}} = \dim_k H^1(k, A_{m,n}; A_{m,n}) = 2S - m - n + 1$. Since $C_{m,n}$ is quasi-homogeneous $\mu = \tau$ ([G]) and the Milnor number of $C_{m,n}$ is also $2S - m - n + 1$. This generalizes the formula of Kouchnirenko.

Let $v_1, \ldots, v_r$, $1 \leq r \leq d$, be the minimal elements of $P-\{0\}$ for a given Λ. Give them indices such that $\nu_1(v_i) < \nu_1(v_{i+1})$ and set $v_0 = m_1$ and $v_{r+1} = m_2$. There exist integers e_i such that $v_{i-1} + v_{i+1} = e_i \cdot v_i$. (The e_i are called the multiplicity of the v_i). One can prove, using only simple vector algebra that

$$\sum_{i=1}^{r} e_i = \delta + r.$$

Laudal and Sletsjøe prove in [L & S] that

$$\tau_X = \dim_k H^1(k, k[\Lambda];\ k[\Lambda]) = \begin{cases} d-1 & \text{if } r=1 \\ \left(\sum_{i=1}^{r} e_i\right) - 2 & \text{if } r>2 \end{cases}$$

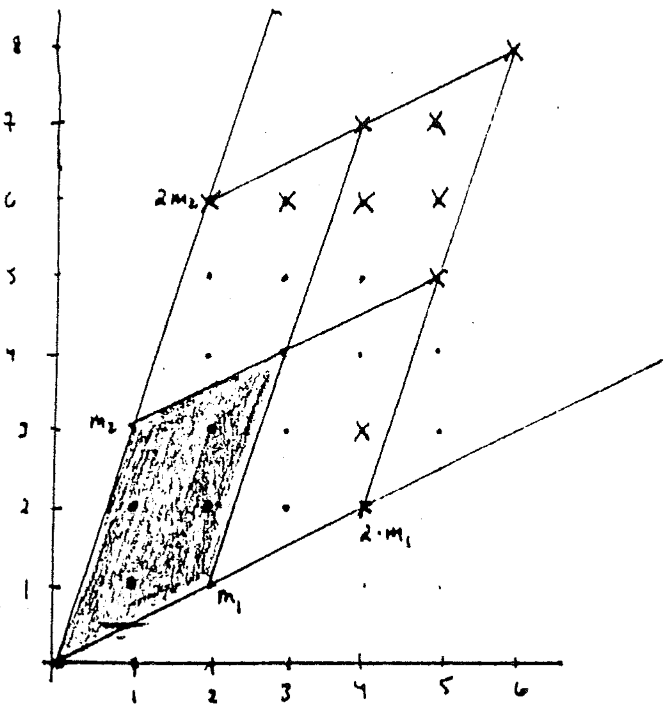
We have thus a formula relating deformations of $C_{m,n}$ on X and deformations of X:

$$\dim_k \mathrm{Def}_{\{C_{m,n} \subseteq X\}}(k[\varepsilon]) = mnd - m - n - \tau_X + r$$

Finally, here are 2 examples. Corollary 1 and 2 have the nice property, that given $k[\Lambda]$ and $f_{m,n}$, we can find the basis and the dimension of $\mathrm{Def}_{\{C_{m,n} \subseteq X\}}(k[\varepsilon])$ by just looking at a diagram.

<u>Example 1</u>

Let $m_1 = (2,1)$, $m_2 = (1,3)$, $m=2$ and $n=2$. We have the following picture



$k[\Lambda] = k[xy^3, xy^2, xy, x^2y]$

$A_{m,n} = k[\Lambda]/(x^4y^2 + x^2y^6)$

The shaded area is P and we see that $d=5$, $\hat{a} = (2,3)$, $\check{a} = (2,2)$, and the points satisfying condition L are: $(2,3)$, $(2,2)$ and $(1,2)$, so $\delta=3$.

The points with an x are the monomials in $Der_{m,n}$.

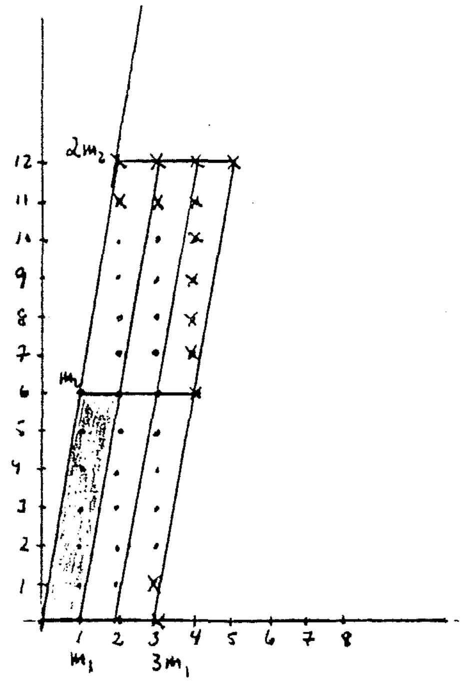
$\dim_k Der_{\{C_{m,n}\subseteq X\}}(k[\varepsilon]) = 2 \cdot 2 \cdot 5 - 2 - 2 - 3 + 2 = 15$ and the basis is

$$\{1,\ xy,\ xy^2,\ xy^3,\ x^2y,\ x^2y^2,\ x^2y^3,\ x^2y^4,\ x^2y^5,$$
$$x^3y^2,\ x^3y^3,\ x^3y^4,\ x^3y^5,\ x^4y^4,\ x^4y^5\}$$

Example 2

Let $m_1 = (1,0)$, $m_2 = (1,6)$, $m=3$ and $n=2$.

$$k[\Lambda]=k[xy^6,\ xy^5,\ xy^4,\ xy^3, xy^2,\ xy,\ x]$$
$$A_{m,n} = k[\Lambda]/(x^2y^{12}+x^3)$$



$d=6$, $\hat{a} = (1,5)$, $\check{a} = (1,1)$. The points satisfying condition L are all the points in $P-\{0\}$, i.e. $(1,1)$, $(1,2)$, $(1,3)$, $(1,4)$ and $(1,5)$, so $\delta=5$

The points x'ed out are in $Der_{m,n}$.

$$\dim_k Der_{\{C_{m,n}\subseteq X\}}(k[\varepsilon]) = 3 \cdot 2 \cdot 6 - 3 - 2 - 5 + 2 = 28$$

References

[B&G] Buchweitz, R.O., Greuel, G. M.: The Milnor Number and Deformations of Complex Curve Singularities, Inv. Math. 58, 241-281 (1980)

[G] Greuel, G.M.: Dualität in der lokalen Kohomologie isolierter Singularitäten, Math.Ann. 250, 157-173 (1980).

[H] Herzog, J: Generators and Relations of Abelian Semigroups and Semigroup Rings. manuscripta math. 3, 175-193 (1970).

[K] Kouchnirenko, A.G.: Polyedres de Newton et nombres de Milnor, Inv. Math 32, 1-31 (1976).

[L] Laudal, O.A.: Formal Moduli of Algebraic Structures, Springer Lecture Notes 754, Berlin 1979.

[L&S] Laudal, O.A., Sletsjøe A.: Cohomology of Groups, Monoids and their Algebras, manuscript

[O] Oda, T.: Lectures on Torus Embeddings and Applications, Bombay, 1978.

[R] Riemenschneider O.: Deformationen von Quotienten singularitäten (nach zyklischen Gruppen), Math. Ann, 211-248 (1974).